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Müntz-Jackson Theorems in L^p, $p < 2$

JOSEPH BAK[1]

Department of Mathematics, City College, New York, N. Y. 10031

AND

DONALD J. NEWMAN[2]

Belfer Graduate School of Science, Yeshiva University, New York, N. Y. 10033

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1. INTRODUCTION

Let $\Lambda = \{1, x^{\lambda_1}, x^{\lambda_2},..., x^{\lambda_n}\}$ where the λ_k are positive numbers satisfying the growth condition $\lambda_k \geqslant 2k$. We seek to estimate the degree of approximation possible to functions in the spaces $L^p[0, 1]$, $1 \leqslant p < 2$, by polynomials in the span $[\Lambda]$ of Λ. To be more precise, we introduce the following sequence of definitions:

$$L^p = L^p[0, 1],$$

$\|f\|_p$ is the usual L^p norm of a function $f \in L^p$,

$$W_p(f; \delta) = \sup_{|h| \leqslant \delta} \|f(x + h) - f(x)\|_p,$$

$$S_p = \{f \in L^p : \|f'\|_p \leqslant 1\},$$

$$I_p = \max_{f \in S_p} \min_{Q \in [\Lambda]} \|f - Q\|_p .$$

In short, S_p represents a class of smooth functions in L^p, and I_p measures the degree of approximation possible to functions in S_p. S_p may be called a "fundamental class," and I_p the L^p approximation index by virtue of the following proposition.

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PROPOSITION. *Suppose $I_p \leqslant \eta$. Then, for any $f \in L^p$, there exists a function $Q \in [\Lambda]$ such that*

$$\| f - Q \|_p \leqslant 2W_p(f; \eta).$$

Proof. See [2].

Our goal, then, is to estimate I_p.

We note first that the analogous problem has been completely solved in the L^p spaces, $2 \leqslant p \leqslant \infty$. ($L^\infty$ denotes the space of continuous functions $C[0, 1]$ with the uniform norm.) The result there is:

THEOREM. *For all p, $2 \leqslant p \leqslant \infty$,*

$$B\epsilon \leqslant I_p \leqslant A\epsilon, \tag{1}$$

where A and B are absolute positive constants and

$$\epsilon = \exp\left(-2 \sum_{k=1}^{n} \frac{1}{\lambda_k}\right).$$

Proof. See [2].

Unfortunately, the problem at hand does not seem to be solvable by any "duality principle." Furthermore, the methods used in [2] involve certain inequalities which are applicable only in the cases $p \geqslant 2$. Nevertheless, our conjecture is that (1) holds for all p; the results contained in this paper show that I_p is, in any case, "roughly speaking" ϵ. We will prove, namely:

THEOREM 1. *For all p, $1 \leqslant p < 2$, $B\epsilon/|\log \epsilon|^{5/2} < I_p \leqslant A\epsilon |\log \epsilon|^{1/p}$, where A and B are absolute positive constants and $\epsilon = (-2 \sum_{k=1}^{n} (1/\lambda_k))$ as before.*

The approach used to obtain the upper bound in Theorem 1 is a combination of estimates contained in [2] and the straightforward evaluation of a critical contour integral. To obtain the lower bound, we use a very elementary and direct approach: we exhibit a function f_p (in fact, a monomial) in each class S_p which cannot be approximated better than the stated lower bound.

2. AN UPPER BOUND FOR I_p

In this section, $\|f\|_q$ will denote the L^q norm on $[0, \infty)$, unless otherwise specified. Let $H \in L^q[0, \infty)$, $q = p/(p-1)$ with $\| H \|_q \leqslant 1$ and such that

$$F(z) = \int_0^\infty e^{-zx} H(x)\, dx = 0 \qquad \text{for} \quad z = \lambda_k + \frac{1}{p}, \quad k = 1, 2, \ldots, n.$$

Define

$$K(t) = \frac{1}{2\pi i} \int_C \frac{e^{zt}F(z)}{z - 1/p} (1 - z^4R^{-4})(1 - p^{-4}R^{-4})^{-1} \, dz,$$

where $C = \{|z| = R: \operatorname{Re} z \geqslant 0\}$, $R = (\epsilon e^{t+1})^{-1}$ and ϵ is as above. The following upper bound was derived in [2]:

For all p, $1 \leqslant p \leqslant \infty$,

$$I_p \leqslant A_1\epsilon + A_2 \sup_K \| e^{-t}K(t)\|_q , \tag{2}$$

where the latter norm may be evaluated on the subinterval $[0, |\log(6\epsilon)|]$. We wish to prove

PROPOSITION 1. *For all p, $1 \leqslant p < 2$, $I_p \leqslant A\epsilon \mid \log \epsilon \mid^{1/p}$.*

By (2), it suffices to show

$$\| e^{-t}K(t)\|_q \quad \text{on} \quad [0, |\log(6\epsilon)|] \leqslant A\epsilon \mid \log \epsilon \mid^{1/p}$$

for some constant A. Towards that end, we record three lemmas, the first two of which were proven in [2].

LEMMA 1. *Let $a_k = \lambda_k + 1/p$, $B(z) = \prod_{k=1}^{n} (a_k - z)/(a_k + z)$, the Blaschke product with zeros a_k. Then $|B(z)| \leqslant 2(e^{3/4}\epsilon \mid z \mid)^{\operatorname{Re} z}$.*

LEMMA 2. *For all $\lambda \geqslant 0$, $C = \{|z| = R, \operatorname{Re} z \geqslant 0\}$*

$$\frac{1}{2\pi} \int_C \left| \frac{e^{-z\lambda}}{z} (1 - z^4R^{-4}) \, dz \right| \leqslant \frac{2}{R^2\lambda^2 + 1} .$$

LEMMA 3. *Again, let $R = (\epsilon e^{t+1})^{-1}$.*

$$\left\| \frac{e^{-t}}{R^2 + 1} \right\|_q \leqslant 6\epsilon.$$

Proof of Lemma 3. While we need only consider $q > 2$, we will prove the lemma for all q, $1 \leqslant q \leqslant \infty$. This will follow from the special cases $q = 1$ and $q = \infty$. For $q = 1$, we have

$$\left\| \frac{e^{-t}}{R^2 + 1} \right\|_1 = \int_0^\infty \frac{e^{-t} \, dt}{(\epsilon e^{t+1})^{-2} + 1} = \int_0^\infty \frac{e^t \, dt}{(e\epsilon)^{-2} + e^{2t}}$$

$$\leqslant \int_{-\infty}^\infty \frac{e^t \, dt}{(e\epsilon)^{-2} + e^{2t}} = \frac{\pi}{2} e\epsilon.$$

For $q = \infty$,

$$\left\| \frac{e^{-t}}{R^2 + 1} \right\|_\infty = \left\| \frac{e^t}{(e\epsilon)^{-2} + e^{2t}} \right\|_\infty = \frac{e\epsilon}{2}$$

by straightforward differentiation. Thus the lemma is proven.

Proof of Proposition 1. We first consider $F(z) = \int_0^\infty e^{-zx} H(x)\, dx$. Let $z = u + iv$, by Hölder's Inequality

$$| F(z)| \leqslant \left(\int_0^\infty e^{-pux}\, dx\right)^{1/p} \leqslant u^{-1/p}.$$

If we restrict ourselves, then, to $\{| z | = R: \operatorname{Re} z \geqslant \delta\}$ and recall

$$F(a_k) = F\left(\lambda_k + \frac{1}{p}\right) = 0, \qquad k = 1, 2,..., n$$

we can use the usual Blaschke estimates to show

$$\left| \frac{F(z)}{B(z)} \right| \leqslant \delta^{-1/p} \Big/ \inf_{\operatorname{Re} z = \delta} | B(z)|, \qquad \text{where} \quad B(z) = \prod_{k=1}^n \frac{a_k - z}{a_k + z}.$$

But clearly

$$\inf | B(z)| = \prod_{k=1}^n \frac{a_k - \delta}{a_k + \delta} = \prod_{k=1}^n \left(1 - \frac{2\delta}{a_k + \delta}\right).$$

By the standard technique equating products of the form $\pi(1 - \alpha_k)$ with exponentials $\exp(-\sum \alpha_k)$, we have

$$\begin{aligned} \inf | B(z)| &\geqslant A_3 \exp\left(-2\delta \sum \frac{1}{a_k + \delta}\right) \\ &\geqslant A_3 \exp\left(-2\delta \sum \frac{1}{\lambda_k}\right) \\ &= A_3 \epsilon^\delta. \end{aligned}$$

Hence

$$| F(z)| \leqslant A_3^{-1} \delta^{-1/p} \epsilon^{-\delta} \,| B(z)|,$$

and by Lemma 1, we have

$$| F(z)| \leqslant A_4 \delta^{-1/p} \epsilon^{-\delta} (e^{3/4} \epsilon \,| z |)^{\operatorname{Re} z}.$$

Setting $\delta = 1/| \log \epsilon |$, we obtain

$$| F(z)| \leqslant A_5 \,| \log \epsilon |^{1/p} \,(e^{3/4} \epsilon \,| z |)^{\operatorname{Re} z} \quad \text{as long as} \quad \operatorname{Re} z \geqslant \frac{1}{| \log \epsilon |}. \tag{3}$$

Finally, we turn to $K(t)$. Clearly,

$$|K(t)| \leqslant \frac{1}{2\pi} \int_C \left| \frac{e^{zt}F(z)}{z - 1/p} (1 - z^4R^{-4})(1 - p^{-4}R^{-4})^{-1}\, dz \right|.$$

Furthermore, since $t < |\log(6\epsilon)|$, $R > 2$ so that

$$|1 - p^{-4}R^{-4}|^{-1} < 2 \qquad \text{and} \qquad \frac{1}{|z - 1/p|} < \frac{2}{|z|}.$$

Hence,

$$|K(t)| < \frac{2}{\pi} \int_C \left| \frac{e^{zt}F(z)}{z} (1 - z^4R^{-4})\, dz \right|.$$

In order to further estimate $K(t)$, we split the contour C into

$$C_1 = \left\{ |z| = R\colon \operatorname{Re} z = u \geqslant \frac{1}{|\log \epsilon|} \right\} \qquad \text{and} \qquad C_2 = C - C_1 .$$

We have, integrating over C_1,

$$\begin{aligned} J_1 &= \int_{C_1} \left| \frac{e^{zt}F(z)}{z} (1 - z^4R^{-4})\, dz \right| \\ &\leqslant A_5 \,|\log \epsilon|^{1/p} \int_{C_1} \left| \frac{(e^{3/4}\epsilon\, |z|)^{\operatorname{Re} z}\, e^{zt}}{z} (1 - z^4R^{-4})\, dz \right| \quad \text{by (3).} \end{aligned}$$

But $|z| = R = (\epsilon e^{t+1})^{-1}$, hence

$$J_1 \leqslant A_5 \,|\log \epsilon|^{1/p} \int_{C_1} \left| \frac{e^{-\frac{1}{4}z}}{z} (1 - z^4R^{-4})\, dz \right|$$

and

$$J_1 \leqslant A_6 \frac{|\log \epsilon|^{1/p}}{R^2 + 1} \qquad \text{by Lemma 2.} \tag{4}$$

Over C_2, we set $z = Re^{i\theta}$ so that $|1 - z^4R^{-4}| = 4 |\sin\theta| \cos\theta$, and we use the fact that $|F(z)| \leqslant u^{-1/p} = |R\cos\theta|^{-1/p}$ to obtain

$$\begin{aligned} J_2 &= \int_{C_2} \left| \frac{e^{zt}F(z)}{z} (1 - z^4R^{-4})\, dz \right| \\ &\leqslant 8 \int_{\theta_1}^{\pi/2} \frac{e^{tR\cos\theta} \cos\theta \sin\theta}{(R\cos\theta)^{1/p}}\, d\theta \qquad \text{with} \quad \theta_1 = \sec^{-1}(|\log \epsilon|\, R) \end{aligned}$$

Setting $\cos\theta = s$,

$$J_2 \leqslant 8 \int_0^{(|\log\epsilon| R)^{-1}} e^{Rts} s (Rs)^{-1/p}\, ds.$$

Now, $Rs < 1/|\log \epsilon|$ and if we reinvoke the condition $t < -\log(6\epsilon) < |\log \epsilon|$, we have $Rts < 1$ and

$$J_2 \leqslant A_7 \int_0^{(|\log\epsilon| R)^{-1}} s^{1/q} R^{-1/p}\, ds.$$

Considering, then, the maximum of the integrand and the length of the interval gives

$$J_2 \leqslant \frac{A_8}{|\log \epsilon|^{1+1/q} R^2} \leqslant A_9 \epsilon^2 e^{2t}. \tag{5}$$

Finally, $|K(t)| \leqslant J_1 + J_2$, so that by (4) and (5), we have

$$|e^{-t}K(t)| \leqslant A_6 \frac{|\log \epsilon|^{1/p} e^{-t}}{R^2 + 1} + A_9 \epsilon^2 e^t.$$

Taking the L^q norm of the above (restricting ourselves to $[0, |\log(6\epsilon)|]$), we have

$$\| e^{-t}K(t)\|_q \leqslant A_6 |\log \epsilon|^{1/p} \left\| \frac{e^{-t}}{R^2 + 1} \right\|_q + A_9 \epsilon^2 \left(\int_0^{|\log 6\epsilon|} e^{qt}\, dt \right)^{1/q}.$$

Hence, by Lemma 3 and direct integration, we have

$$\begin{aligned} \| e^{-t}K(t)\|_q &\leqslant A_{10}[|\log \epsilon|^{1/p} \epsilon + \epsilon] \\ &\leqslant A\epsilon |\log \epsilon|^{1/p}, \end{aligned}$$

and the proof is complete.

3. A Lower Bound for I_p

Throughout this section, we will find it necessary to modify Λ by translating the exponents or adding a single monomial. Hence, we introduce the following notation:

$$\Lambda_a = \{1, x^{\lambda_1 + a}, x^{\lambda_2 + a}, ..., x^{\lambda_n + a}\},$$
$$\Lambda_a^{\lambda} = \{1, x^{\lambda}, x^{\lambda_1 + a}, ..., x^{\lambda_n + a}\}.$$

We also define $d_p(f, \Lambda)$ to be the L^p distance of the function f to the space $[\Lambda]$:

$$d_p(f, \Lambda) = \inf_{Q \in [\Lambda]} \| f - Q \|_p,$$

where $\| \ \|_p$ here and throughout the rest of the paper will denote the usual L^p norm on $[0, 1]$. Using the above notation, we will prove the following key lemmas.

LEMMA 1. *Suppose* $0 < \delta < \frac{1}{5}$. *Then there exist positive constants* A_1 *and* A_2 *such that*

$$d_2(x^{1/2+\delta}, \Lambda) \geqslant A_1 \epsilon^{1+\delta} \tag{A}$$

$$d_2(x^{1/2+\delta}, \Lambda_a^{\lambda}) \geqslant A_2 \delta \epsilon^{1+\delta} \quad \text{as long as} \quad a \geqslant -1, \quad |\lambda - \tfrac{1}{2} - \delta| > \delta. \tag{B}$$

LEMMA 2. *Let* $a \geqslant 0$, $\alpha = 1/|\log \epsilon|$. *Then there exists* $A > 0$ *such that*

$$d_\infty(x^{1+\alpha}, \Lambda_a^b) \geqslant A \frac{\epsilon}{|\log \epsilon|^{3/2}} \quad \text{as long as} \quad |b - 1 - \alpha| \geqslant \alpha.$$

Proof of Lemma 1. First of all, an exact formula for $d_2(x^N, \Lambda)$ is given by

$$d_2(x^N, \Lambda) = \frac{N}{(N+1)\sqrt{2N+1}} \prod_{k=1}^{n} \left| \frac{\lambda_k - N}{\lambda_k + N + 1} \right| \quad \text{(e.g., see [1], p. 20).} \tag{6}$$

Setting $N = \frac{1}{2} + \delta$ and replacing the above product with the appropriate exponential, we have

$$\begin{aligned} d_2(x^{1/2+\delta}, \Lambda) &\geqslant A_1 \exp\left(-2 \sum (1+\delta)/(\lambda_k + \tfrac{3}{2} + \delta)\right) \\ &\geqslant A_2 \epsilon^{1+\delta}. \end{aligned}$$

This proves (A). Furthermore, considering (6) once again with $N = \frac{1}{2} + \delta$ and translating the λ_k by a, we see that the only possible smaller factor introduced is

$$\frac{\lambda_1 + a - \frac{1}{2} - \delta}{\lambda_1 + a + \frac{3}{2} + \delta}.$$

But $\lambda_1 \geqslant 2$, $a \geqslant -1$ and $\delta < \frac{1}{5}$, hence

$$\frac{\lambda_1 + a - \frac{1}{2} - \delta}{\lambda_1 + a + \frac{3}{2} + \delta} \geqslant \frac{\frac{1}{2} - \delta}{\frac{3}{2} + \delta} \geqslant \frac{1}{10}.$$

Finally, adding the single monomial x^λ to Λ introduces a factor of

$$\frac{\lambda - \frac{1}{2} - \delta}{\lambda + \frac{3}{2} + \delta} \geqslant \delta$$

by hypothesis. Hence the lemma is proven.

Proof of Lemma 2. Suppose $\| x^{1+\delta} - Q(x) \|_\infty \leqslant m$ where $Q(x) \in \Lambda_a^b$, $a \geqslant 0$, $|b - 1 - \delta| \geqslant \delta$. Then

$$I = \int_0^1 |x^{1+\delta} - Q(x)|^2 \frac{dx}{x^{1-2\delta}} \leqslant \frac{m^2}{2\delta}.$$

But

$$I = \int_0^1 (x^{1/2+2\delta} - Q^*(x))^2\,dx \geqslant [d_2(x^{1/2+2\delta}, \Lambda_{a+\delta-1/2}^{b+\delta-1/2})]^2$$
$$\geqslant A\delta^2\epsilon^{2+4\delta} \quad \text{by Lemma 1.}$$

Hence,

$$m \geqslant \delta^{3/2}\epsilon^{1+2\delta} \qquad \text{and setting} \quad \delta = \alpha = \frac{1}{|\log \epsilon|}$$

gives the result.

For later purposes, we note that setting $\delta = 2\alpha$ would yield

$$d_\infty(x^{1+2\alpha}, \Lambda_a{}^b) \geqslant A\,\frac{\epsilon}{|\log \epsilon|^{3/2}}$$

as long as the appropriate condition $|b - 1 - 2\alpha| \geqslant \alpha$ is satisfied.

We are now ready to prove

PROPOSITION 2.

$$I_1 \geqslant A\,\frac{\epsilon}{|\log \epsilon|^{3/2}}. \tag{A'}$$

$$\textit{For all } p \geqslant 1, \quad I_p \geqslant A\,\frac{\epsilon}{|\log \epsilon|^{5/2}}. \tag{B'}$$

Proof of (A′).

Let $\alpha = 1/|\log \epsilon|$ as above. We use the fact that $x^\alpha \in S_1$ and hence $I_1 \geqslant d_1(x^\alpha, \Lambda)$.

Suppose then that $\| x^\alpha - p(x)\|_1 \leqslant m$. Let $I(x) = |\int_0^x [t^\alpha - p(t)]\,dt|$, then for all $x \in [0, 1]$, $I(x) \leqslant \int_0^1 |t^\alpha - p(t)|\,dt \leqslant m$. But for some x,

$$I(x) = |x^{1+\alpha} - Q(x)| \geqslant d_\infty(x^{1+\alpha}, \Lambda_1{}^1)$$
$$\geqslant A\,\frac{\epsilon}{|\log \epsilon|^{3/2}} \quad \text{by Lemma 2.}$$

A consideration of the two inequalities, then, proves (A′).

Proof of (B′).

Here we use the fact that $f_p = \frac{1}{2}\alpha^{1/p}x^{1/q+\alpha} \in S_p$, where α is as before and $q = p/(p-1)$ is the conjugate of p. Let us assume then that

$$\| x^{1/q+\alpha} - Q(x)\|_p \leqslant m.$$

Then

$$I(x) = \left| \int_0^x \frac{[t^{1/q+\alpha} - Q(t)]}{t^{1/q-\alpha}} \, dt \right| \leqslant \| t^{1/q+\alpha} - Q \|_p \cdot \| t^{\alpha-1/q} \|_q$$

by Hölder's Inequality

$$\leqslant m \cdot \alpha^{-1/q}.$$

But for some x,

$$I(x) = \left| \frac{x^{1+2\alpha} - Q^*(x)}{1 + 2\alpha} \right| \geqslant \tfrac{1}{2} d_\infty(x^{1+2\alpha}, \Lambda_{\alpha+1/p}^{\alpha+1/p}).$$

By the note following Lemma 2, then, we have

$$m \geqslant A\alpha^{1/q} \frac{\epsilon}{|\log \epsilon|^{3/2}}.$$

Since $d_p(f_p, \Lambda) = (\alpha^{1/p}/2)\, d_p(x^{1/q+\alpha}, \Lambda)$ we have

$$d_p(f_p, \Lambda) \geqslant A\alpha^{1/p+1/q} \frac{\epsilon}{|\log \epsilon|^{3/2}} = A \frac{\epsilon}{|\log \epsilon|^{5/2}}$$

and the proof is complete.

References

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